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Knotted Wave Dislocation with the Hopf Invariant

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We study the wave dislocations with an induced gauge potential. The topological current characterized the wave dislocations is constructed with the dual of Abelian gauge field. And the topological charges and locations of the wave dislocations are determined by the ϕ-mapping topological current theory. Furthermore, it is shown that the knotted wave dislocations can be described with a Hopf invariant in the wave field. At last we discussed the evolution of the knotted wave dislocations.

KEY WORDS: wave dislocation; knot; Hopf invariant.

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1. INTRODUCTION

Wave dislocations (phase singularities) have drawn great interest because they are of importance for understanding fundamental physics and have many important applications. They appear in different areas of both classical and quantum physics, including condensed matter, fluid dynamics, superconductivity, acoustic and optics (Berry, 1981; Bialynicki-Birula *et al.*, 2000; Dennis, 2001; Nye and Berry, 1974). Besides their fundamental far-reaching importance, wave front dislocations have potential practical applications in fields as diverse as oceangraphy, information

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processing, or biology (Ashkin *et al.*, 1987; He *et al.*, 1995; Luther-Davies *et al.*, 1997; Padgett and Allen, 1997; Rozas *et al.*, 1997). Berry and coauthors (Berry, 1976; Berry and Hannay, 1977; Berry *et al.*, 1979; Berry and Wright, 1980; Nye, 1979, 1981, 1983, 1998) considered the scalar wave equation and proved the existence of wave dislocations by exhibiting a number of special solutions of the scalar wave equation that have the dislocation properties. Wave front dislocations appear as a phase singularity where the phase of the wave is undefined thus its amplitude must vanish. The order of the singularity multiplied by its sign is referred to as the topological charge of the dislocation.

From the viewpoint of the Fu *et al.* (2000) and Duan *et al.* (2002), the vortex lines can be studied with the Abelian gauge field. The vortex current characterized with the winding number is the dual of the Abelian gauge field strength. In the different systems the Abelian gauge potential emerge with the different form, e.g., electromagnetic gauge potential in the superconductor (Duan *et al.*, 2002) and the superfluid velocity $\mathbf{v}_s = (\hbar/m_4)\nabla S$ in the superfluid ^{4}He (Thouless, 1998). For the optical field, the "induced Abelian gauge potential" A_μ (Bohm *et al.*, 1991; Holz, 1991, 1992; Hsiang and Lee, 2001) can be constructed with the complex scalar field $\phi(x)$. Thereby we will discuss the wave dislocations from the view of the gauge field.

On the other hand, it is long time to know that the dislocations can form the closed lines (Berry, 1981; Nye and Berry, 1974), which bring us to consider the knotted dislocation lines. A recent study of the knotted wave dislocations governed by Helmholtz equations revealed the universal topological features of dislocation loops. The exact solutions of the wave equation were constructed to represent these closed dislocation lines (Berry and Dennis, 2001a,b).

In this paper, we study the topological properties of dislocation lines from the view of gauge field similar to the field description of dislocations in the crystals (Duan and Zhang, 1990, 1991a,b). By virtue of the well-known "induced gauge potential" the topological current of complex scalar wave is constructed to represent dislocation lines that are straight or curved, or form closed knots. Furthermore, the knotted wave dislocations are discussed with the Hopf invariant in terms of the ϕ-mapping topological current theory. With the decomposition of $U(1)$ gauge potential, we concluded that the Hopf invariant is related to the linking and self-linking number of the knotted dislocation lines.

This paper is arranged as follows. In the Section 2, we give the inner structure of wave dislocation lines with the wave field $\phi(x)$ from the view of Abelian gauge field. Then the knotted dislocation lines are discussed with the Hopf invariant and the ϕ-mapping topological current theory in the Section 3. Furthermore, in terms of the bifurcation theory of topological current we studied the evolution of knotted dislocation lines in Section 4. At last, our conclusion is presented in Section 5.

2. WAVE DISLOCATION WITH GAUGE FIELD

The field including wave dislocations is described by the complex scalar wave solutions $\phi(r)$ of the Helmholtz equation

$$\nabla^2\phi(r) + \phi(r) = 0.$$

Usually the dislocations can be presented by solving the above wave equation. In this paper, we try to discuss the wave dislocation from the viewpoint of gauge field or pure geometry language since the wave dislocations are topological objects on wave-front surfaces.

Let us focus on the complex scalar wave $\phi(r) = \rho(r)n(r)$ ($\phi^*\phi = \rho^2$) associated with the dislocations. The well-known "induced gauge potential" can be constructed with (Bohm *et al.*, 1991; Holz, 1991, 1992; Hsiang and Lee, 2001)

$$A_i = \frac{1}{i}\langle n|\partial_i n\rangle, \quad i = 1, 2, 3. \tag{1}$$

Since wave fields admit a $U(1)$ group of transformations

$$n(r) \to e^{i\theta}n(r),$$

we introduce a "gauge transformation" on A_i,

$$A_i \to A_i + \partial_i\theta.$$

One can find that the above gauge potential has the same form as the decomposed $U(1)$ connection (Duan *et al.*, 1994, 2002; Fu *et al.*, 2000), which is very useful to discuss the vortices of Bose–Einstein condensate (Duan *et al.*, 2003) and cosmic strings in the early university (Duan *et al.*, 1997; Zhang *et al.*, 2003).

The topological current of the wave fields $\phi(r) = \rho(r)(n^1(r) + in^2(r))$ is proposed with the gauge field $F_{ij} = \partial_i A_j - \partial_j A_i$:

$$J^i = \frac{1}{4\pi}\varepsilon^{ijk}F_{ij} = \frac{1}{2\pi}\varepsilon^{ijk}\varepsilon_{ab}\partial_j n^a\partial_k n^b, \tag{2}$$

then one can obtain (Duan and Ge, 1979; Duan *et al.*, 1998)

$$J^i = \delta(\vec{\phi})D^i\left(\frac{\phi}{x}\right), \tag{3}$$

where $D^i(\phi/x) = \frac{1}{2}\varepsilon^{ijk}\varepsilon_{ab}\partial_j\phi^a\partial_k\phi^b$ is the Jacobian vector and $\vec{\phi} = (\phi^1, \phi^2)$ can be regarded as a two-dimensional vector field. It is obvious that the dislocation lines always appear in the zeroes of the $\vec{\phi}(x)$, so it is necessary to study the zero points of $\vec{\phi}(x)$ to determine the non-zero solutions of J^i. One can find that the general solutions of $\vec{\phi}(x, y, z) = 0$ can be expressed as $x = x_l^1(s)$, $y = y_l^2(s)$, $z = z_l^3(s)$ ($l = 1, \ldots, N$) under the regular condition $D^i(\phi/x) \neq 0$, which represent the world surfaces of N isolated singular strings L_l with string parameter s. These singular string solutions are just the dislocation lines and the location of lth dislocation lines is determined by the lth zero point $\vec{r}_l(s)$.

With δ-function theory (Schouten, 1951) and ϕ-mapping topological current theory, one can prove that (Duan and Ge, 1979; Duan *et al.*, 1998)

$$J^i = \sum_{l=1}^{N} \beta_l \eta_l \int_{L_l} \frac{dx^i}{ds} \delta^3(\vec{r} - \vec{r}_l(s)) ds \tag{4}$$

where $\eta_l = sgn D(\phi/u)_{\vec{x}_l} = \pm 1$ is the Brouwer degree of ϕ-mapping, β_l is the Hopf index of ϕ-mapping (Milnor, 1966), which means that when $\vec{r}$ covers the neighborhood of the zero point $\vec{r}_l(s)$ once, the field $\vec{\phi}(x)$ covers the corresponding region in ϕ-space β_l times. Equation (4) gives the topological structure of dislocation lines. Let Σ_l is the lth planer element transversal to L_l, then the topological charge of lth dislocation line L_l is

$$Q_l = \int_{\Sigma_l} J^i d\sigma_i = W_l, \tag{5}$$

where $W_l = \beta_l \eta_l$ is the winding number of $\vec{\phi}$ around L_l. Furthermore, the evolution of wave dislocation lines, for example production, annihilation, splitting and emerging etc, can be discussed with this topological current (Duan *et al.*, 1999a,b). Here one can notice that the sign of topological charge Q_l is determined with Brouwer degree η_l, which is consistent with the definition of Freund (Freund, 2000a,b; Freund and Kessler, 2001).

3. KNOTTED WAVE DISLOCATION WITH THE HOPF INVARIANT

As a kind of curves in space, it is natural to consider the closed and knotted wave dislocation lines in the wave field (Berry, 1981; Berry and Dennis, 2001a,b; Nye and Berry, 1974). Berry *et al.* constructed the exact solutions of the Helmholtz equation to represent the knotted dislocation lines (Berry and Dennis, 2001a,b). In this paper, we try to study the knotted dislocation lines with the Hopf invariant (Duan *et al.*, 2003; Kundu and Rybakov, 1982; Moffatt, 1969; Moffatt and Ricca, 1992; Poenaru and Toulouse, 1977; Winfree and Strogatz, 1983a,b,c) instead of solving the concrete Helmholtz equation. For a closed three-manifold M the Hopf invariant H is given by

$$H = \frac{1}{2\pi} \int_M \varepsilon^{ijk} A_i \partial_j A_k. \tag{6}$$

With the topological structure of the dislocation lines (4) we find that

$$H = \int A_i J^i d^3x = \sum_{l=1}^{N} W_l \int_{L_l} A_i dx^i. \tag{7}$$

It can be seen that when these N dislocation lines are N closed curves, i.e., a family of N knots γ_l ($l = 1, \ldots, N$), (7) means that Hopf invariant describes the knotted dislocations and is a topological invariant (Eguchi *et al.*, 1980).

In the following we investigate the integral (7) in detail. One can find that the Eq. (7) can be expressed as

$$H = \sum_{k,\,l=1}^{N} W_k W_l \oint_{\gamma_k} \oint_{\gamma_l} \partial_i A_j dx^i dy^j,$$

where $\vec{x}$ and $\vec{y}$ are two points on knots γ_k and γ_l, respectively. There are three parts for the above sum: $\vec{x} = \vec{y}$ and $k = l$ (same knot); $\vec{x} \neq \vec{y}$ and $k = l$; $\vec{x} \neq \vec{y}$ and $k \neq l$ (different knots). Defining a 3-dimensional unit vector $\vec{m} = (\vec{y} - \vec{x})/||\vec{y} - \vec{x}||$, which form a sphere S^2, then one obtain a 2-dimensional vector field $\vec{e}$ with $2\varepsilon_{ab}\partial_i e^a \partial_j e^b = \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m})$ (Mermin and Ho, 1976; Morandi, 1992). The induced gauge potential A_i can be decomposed with this 2-dimensional unit vector (Duan *et al.*, 1994, 2002; Fu *et al.*, 2000)

$$A_i = \varepsilon_{ab} e^a \partial_i e^b. \quad (a, b = 1, 2)$$

Using above expression, the Hopf invariant can be expressed as

$$H = \sum_{k=1}^{N} W_k^2 \oint_{\gamma_k} \epsilon_{ab} e^a \partial_i e^b dx^i + \sum_{k=1\ (\vec{x}\neq\vec{y})}^{N} \frac{1}{2} W_k^2 \oint_{\gamma_k} \oint_{\gamma_k} \vec{m}^*(dS)$$
$$+ \sum_{k,l=1\ (k\neq l)}^{N} \frac{1}{2} W_k W_l \oint_{\gamma_k} \oint_{\gamma_l} \vec{m}^*(dS), \tag{8}$$

where $\vec{m}^*(dS) = \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) dx^i \wedge dy^j$ $(\vec{x} \neq \vec{y})$ denotes the pull-back of the S^2 surface element.

In the following we will investigate the three terms in the r.h.s of Eq. (8) in detail. The first term of (8) is just related to the twisting number $Tw(\gamma_k)$ of γ_k

$$\frac{1}{2\pi} \oint_{\gamma_k} \epsilon_{ab} e^a \partial_i e^b dx^i = \frac{1}{2\pi} \oint_{\gamma_k} (\vec{T} \times \vec{V}) \cdot d\vec{V} = Tw(\gamma_k), \tag{9}$$

where $\vec{T}$ is the unit tangent vector of knot γ_k at $\vec{x}$ ($\vec{m} = \vec{T}$ when $\vec{x} = \vec{y}$), and $\vec{V}$ is defined as $e^a = \epsilon^{ab} V^b$ ($\vec{V} \perp \vec{T}$, $\vec{e} = \vec{T} \times \vec{V}$). For the second term, one can prove that it is related to the writhing number $Wr(\gamma_k)$ of γ_k (Calini and Ivey, 1996; Pohl, 1968; Rolfsen, 1976)

$$Wr(\gamma_k) = \frac{1}{4\pi} \oint_{\gamma_k} \oint_{\gamma_k} \vec{m}^*(dS). \tag{10}$$

In terms of the Calugareanu formula (Calini and Ivey, 1996; Pohl, 1968; Rolfsen, 1976)

$$SL(\gamma_k) = Wr(\gamma_k) + Tw(\gamma_k), \tag{11}$$

we see that the first and the second terms of (8) just compose the self-linking numbers of knots. Furthermore, for the third term, one can prove

$$\frac{1}{4\pi}\oint_{\gamma_k}\oint_{\gamma_l}\vec{m}^*(dS) = \frac{1}{4\pi}\epsilon^{ijk}\oint_{\gamma_k}dx^i\oint_{\gamma_l}dy^j\frac{(x^k - y^k)}{\|\vec{x}-\vec{y}\|^3} = Lk(\gamma_k,\gamma_l)\ \ (k \neq l), \tag{12}$$

where $Lk(\gamma_k, \gamma_l)$ is the Gauss linking number between γ_k and γ_l, (Witten, 1989; Polyakov, 1988). Therefore, from (10), (9), (11) and (12) we obtain the important result:

$$H = 2\pi\left[\sum_{k=1}^{N} W_k^2 SL(\gamma_k) + \sum_{k,l=1\ (k\neq l)}^{N} W_k W_l Lk(\gamma_k,\gamma_l)\right]. \tag{13}$$

This precise expression just reveals the relationship between H and the self-linking and the linking numbers of the knots family. Since the self-linking and the linking numbers are both the invariant characteristic numbers of the knots family in topology, H is an important invariant required to describe the knotted dislocation lines in the wave field.

4. EVOLUTION OF KNOTTED DISLOCATION LINES

The evolution of wave dislocation is of considerable interest. In the above section, we did not consider the motion of dislocation line L, and only discussed the space structure of dislocation lines in the three-dimensional space. In this section, we will investigate the evolution of the dislocation lines in (3 + 1)-dimensional spacetime with coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, and $x^0 = t$. Here we fix the x^3 coordinate for simplicity and take the XOY plane as the cross section, so the intersection line between the L's evolution surface and the cross section is just the motion curve of L (Duan and Zhang, 1999). In this case the 2-dimensional topological current have the form with the coordinate t instead of z

$$J^i = \delta^2(\phi)D^i(\phi/x), \quad (i = 1, 2)$$

and the density of dislocation lines

$$\rho = \delta^2(\phi)D^0(\phi/x), \tag{14}$$

where $D^1(\phi/x) = \varepsilon_{ab}\partial_2\phi^a\partial_0\phi^b$, $D^2(\phi/x) = \varepsilon_{ab}\partial_0\phi^a\partial_1\phi^b$ and $D^0(\phi/x) = \varepsilon_{ab}\partial_1\phi^a\partial_2\phi^b$. From Eq. (14) the line density of line defects on the cross section do not vanish only at the zero points of vector order parameter $\vec{\phi}(x, y, t)$, i.e.,

$$\phi^1(x, y, t) = 0, \quad \phi^2(x, y, t) = 0, \tag{15}$$

which determine the positions of line defects. If the Jacobian determinant $D^0(\phi/x) \neq 0$, the solutions of Eq. (15) are expressed as

$$x = x_l(t), \quad y = y_l(t), \quad l = 1, 2, \ldots, N \tag{16}$$

which represent the motion curves of N zero point $\vec{x}_l(t)$ on the cross section, and which show them moving in (2 + 1)-dimensional space-time. In our previous work (Duan and Zhang, 1999) it has been pointed out that, during the evolution of line defects, when the regular condition $D^0(\phi/x) \neq 0$ fails, the branch processes (i.e. the splitting, mergence and intersection) will occur; and in these branch processes, the sum of the topological charges of final line defect(s) W_f is equal to that of the initial line defect(s) W_i at the bifurcation point.

In following we will show that when the branch processes of knotlike dislocation lines occur, the topological invariant H of (8) (i.e. (13)) is preserved:

(i) The splitting case. We will consider one knot γ split into two knots γ_1 and γ_2 which are of the same self-linking number as γ $(SL(\gamma) = SL(\gamma_1) = SL(\gamma_2))$, and will compare the two numbers H_γ and $H_{\gamma_1+\gamma_2}$, where H_γ is the contribution of γ to H before splitting, and $H_{\gamma_1+\gamma_2}$ is the total contribution of γ_1 and γ_2 to H after splitting. Firstly, from the above text we have $W_\gamma = W_{\gamma_1} + W_{\gamma_2}$ in the splitting process. Secondly, on the one hand, noticing that in the neighborhood of bifurcation point, γ_1 and γ_2 are infinitesimally displaced from each other; on the other hand, for a knot γ its self-linking number $SL(\gamma)$ is defined as $SL(\gamma) = Lk(\gamma,\ \gamma_V)$, where γ_V is another knot obtained by infinitesimally displacing γ in the normal direction $\vec{V}$ (Witten, 1989). Therefore $SL(\gamma) = SL(\gamma_1) = SL(\gamma_2) = Lk(\gamma_1,\ \gamma_2) = Lk(\gamma_2,\ \gamma_1)$, and $Lk(\gamma,\ \gamma_k') = Lk(\gamma_1,\ \gamma_k') = Lk(\gamma_2,\ \gamma_k')$ (here γ_k' denotes another arbitrary knot in the family $(\gamma_k' \neq \gamma,\ \gamma_k' \neq \gamma_{1,2})$). Then, thirdly, we can compare H_γ and $H_{\gamma_1+\gamma_2}$ as: before splitting, from (13) we have

$$H_\gamma = 2\pi\left[W_\gamma^2 SL(\gamma) + \sum_{k=1\ (\gamma_k'\neq\gamma)}^{N} 2W_\gamma W_{\gamma_k'} Lk(\gamma, \gamma_k')\right], \tag{17}$$

where $Lk(\gamma, \gamma_k') = Lk(\gamma_k', \gamma)$; after splitting,

$$\begin{aligned} H_{\gamma_1+\gamma_2} = 2\pi\Bigg[&W_{\gamma_1}^2 SL(\gamma_1) + W_{\gamma_2}^2 SL(\gamma_2) + 2W_{\gamma_1} W_{\gamma_2} Lk(\gamma_1, \gamma_2) \\ &+ \sum_{k=1\ (\gamma_k'\neq\gamma_{1,2})}^{N} 2W_{\gamma_1} W_{\gamma_k'} Lk(\gamma_1, \gamma_k') + \sum_{k=1\ (\gamma_k'\neq\gamma_{1,2})}^{N} 2W_{\gamma_2} W_{\gamma_k'} Lk(\gamma_2, \gamma_k')\Bigg]. \end{aligned} \tag{18}$$

Comparing (17) and (18), we just have

$$H_\gamma = H_{\gamma_1+\gamma_2}. \tag{19}$$

This means that in the splitting process H is preserved.

(ii) The mergence case. We consider two knots γ_1 and γ_2, which are of the same self-linking number, merge into one knot γ which is of the same self-linking number as γ_1 and γ_2. This is obviously the inverse process of the above splitting case, therefore we have

$$H_{\gamma_1+\gamma_2} = H_\gamma. \tag{20}$$

(iii) The intersection case. This case is related to the collision of two knots (Niemi, 2000). We consider two knots γ_1 and γ_2, which are of the same self-linking number, meet, and then depart as other two knots γ_3 and γ_4 which are of the same self-linking number as γ_1 and γ_2. This process can be identified to two sub-processes: γ_1 and γ_2 merge into one knot γ, and then γ split into γ_3 and γ_4. Thus from the above two cases (ii) and (i) we have

$$H_{\gamma_1+\gamma_2} = H_{\gamma_3+\gamma_4}. \tag{21}$$

Therefore we obtain the result that, in the branch processes during the evolution of knotlike dislocations (i.e., the splitting, mergence and intersection), the topological invariant H is preserved.

5. CONCLUSION

In this paper, we investigate the topological properties of wave dislocation lines in terms of the ϕ-mapping topological current theory. First, the dislocation lines are studied with the "induced gauge field" and it is shown that the dislocation lines originated from the singularities of wave field. The topological structure of the wave dislocation lines is obtained based on the δ-function theory. Here we want to point out that these topological structures can all be characterized by the ϕ-mapping topological numbers–Hopf indices and Brouwer degrees, and their locations can be rigorously determined by ϕ-mapping topological current theory. Secondly, with the help of Hopf invariant we discussed the knotted dislocations and show that the linking and self-linking number of the knotted dislocation is just the Hopf invariant. At last, it is shown that the topological number H is preserved in the branch processes (splitting, mergence and intersection) during the evolution of these knotted dislocation lines.

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